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Comment on the Interpretation of Inductive Probabilities

Abner Shimony[1]

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A recent defense of Jaynes' information-theoretical approach to statistical mechanics is rejected, and an earlier critique of this approach is extended.

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A recent paper by Hobson[(1)] defends Jaynes' information-theoretical approach to statistical mechanics against a criticism by Friedman and Shimony (FS).[(2)] Friedman and Shimony showed that Jaynes' use of the concept of inductive probability, in conjunction with his maximum entropy principle, leads in a rather special situation to a conclusion which they regarded as unacceptable, namely $\rho(D_F \mid B) = \delta(F - \langle f \rangle)$, where D_F is the proposition that the random variable $\bar{f} = \lim_{n\to\infty}(1/n) \sum_{k=1}^{n} f(i_k)$ has the value F, $\langle f \rangle$ is the expectation value of the random variable f on the background data B, and $\rho(D_F|B)$ is the probability distribution of D_F on background data B. Hobson presents a theorem to the effect that the same conclusion holds in quite a wide class of situations. More important, he argues that the conclusion in question is reasonable.

Hobson dispenses with the assumption made by FS that f has a certain special form. However, he states an assumption, not made by FS, that "the data B are symmetric with respect to different trials" and "do not link the

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[1] Laboratoire de Physique Théorique et Particules Elémentaires, Bâtiment 211, Université Paris-Sud, Orsay, France. Permanent address: Departments of Philosophy and Physics, Boston University, Boston, Massachusetts.

outcome of one trial to the outcome of any other." Hobson's theorem also depends upon an application of Jaynes' maximum entropy principle to a sample space $\mathscr{S}^n$ in which the points are the joint outcomes of n trials, each trial having one of a countable set of outcomes. (He uses the formula $P(i_1 i_2 \cdots i_n \mid B) = \prod_{k=1}^{n} P(i_k \mid B)$, for which he refers to his book,[3] and one can easily see by examining pp. 49, 78, and 163 that he applies the maximum entropy principle to $\mathscr{S}^n$. An advocate of the information-theoretical approach to statistical mechanics who finds the conclusion in question unacceptable could, conceivably, escape from Hobson's theorem by denying the applicability of Jaynes' principle to $\mathscr{S}^n$. The advocate could not escape from the argument of FS in this way, since FS applied the principle only to the space $\mathscr{S}$ of outcomes from a single trial, following examples in Jaynes' own papers. Nevertheless, Hobson's theorem, which mathematically is surely correct, is valuable, for even if the conclusion is unreasonable, the theorem may be useful in helping to locate questionable assumptions which are implicit in the maximum entropy principle.

As to the reasonableness of the conclusion, Hobson argues as follows. He admits that there exist data B' such that $\rho(D_F \mid B')$ is not a delta function concentrated at $\langle f \rangle$. Indeed, if B' assert, for example, that a die is weighted heavily near one face, then $\rho(D_F \mid B')$ is close to a delta function concentrated at a value different from $\langle f \rangle$. His theorem is not contradicted, however, because the data B' assert a linkage among the trials. The prediction based upon B' is irrelevant to the question at issue, since the data B' are not included in B. "Thus, the resolution of the difficulty is simply that inductive predictions, even when they are 'certain', may turn out to be wrong if the data on which they were based are incomplete in some important respect." ('Certain' in quotation marks signifies a probability of one relative to the data.)

An evaluation of this argument would be expedited by a fuller explanation of the expression "the data B do not link ..." than Hobson provides, but his intension is somewhat indicated by one of his examples. He lets B be the information that the successive trials are the results of tosses of a single die which is described only as being cubical and as having i spots on the ith side. (Presumably, he would also allow the background data B to include information about such general matters as the laws of physics, the existence of a gravitational field near the surface of the earth, and the procedures for tossing dice.) The outcomes of the various trials are surely linked in the sense that a single die is used throughout, so that any weighting which influences the outcome of one toss also influences the outcomes of the others. Nevertheless, Hobson apparently considers B in this case to satisfy his condition of "not linking." His theorem, therefore, is applicable and yields the conclusion $\rho(D_F \mid B) = \delta(F - \langle i \rangle)$, where the random variable f has been taken to be i (the number of spots showing) and where $\langle i \rangle = 3.5$ as a consequence of

$P(i \mid B) = 1/6$. To evaluate this conclusion, consider the family of propositions $\{W_{\mathbf{x}}\}$, where $\mathbf{x}$ is a vector from the center of the die to an arbitrary point within it, and $W_{\mathbf{x}}$ asserts that the center of mass of the die is located at $\mathbf{x}$. Now, B logically implies that one of the $W_{\mathbf{x}}$ is true. Hence

$$\rho(D_F \mid B) = \int_V \rho(D_F \mid B \,\&\, W_{\mathbf{x}})\rho(W_{\mathbf{x}} \mid B)\, d^3\mathbf{x}$$

Now partition the region V occupied by the die into two subregions V_1 and V_2, such that $\rho(D_F \mid B \,\&\, W_{\mathbf{x}}) = \delta(F - \langle i \rangle)$ for $\mathbf{x} \in V_1$, and this equation fails for $x \in V_2$. Hobson's conclusion requires that $\int_{V_2} \rho(W_{\mathbf{x}} \mid B)\, d^3\mathbf{x} = 0$. But it is hard to see how this could be reasonable, even from Jaynes' point of view. Once it is granted that a value of x somewhat displaced from the center of the die toward a face favors the appearance of the opposite face, as Hobson seems to grant in his remarks on weighting, then the volume of V_2 is a large part of the volume of V. (Indeed, if one takes the delta function literally, then V_1 may be a set of measure zero.) Reasonableness may not require that the distribution $\rho(W_{\mathbf{x}} \mid B)$ be uniform over V, but it surely does not permit any large subregion of V to be assigned probability zero. In particular, Jaynes' point of view, which insists upon acknowledging ignorance, would surely not assign probability zero to V_2 upon data B. Hence, Hobson's conclusion, $\rho(D_F \mid B) = \delta(F - \langle i \rangle)$, should be denied.

It is possible that Hobson would wish to maintain his theorem but to retract his tacit assent to the premiss that "the data B do not link..." is satisfied in the example of the die. This would be a reasonable move, since, as remarked above, the employment of a single physical object in successive trials does constitute a kind of linkage. However, this move would not save Jaynes' program from the difficulty presented by FS, who do not make the assumption of "not linking." The analysis of FS does not apply to a normal die if $f(i)$ is taken to be i, since FS suppose that there is a possible value of the random variable f which equals $\langle f \rangle$, but $\langle i \rangle = 3.5$ is not equal to the number of spots on any face. However, the result proved in the appendix of this paper is applicable, because it dispenses with this supposition. Alternatively, one can slightly modify the problem so that the FS analysis does apply, by using the random variable $f(i) = i$ for $i = 1, 2, 3, 4, 5$ and $f(6) = 9$. Then $\langle f \rangle = 4$, which is one of the values of f. The conclusion of the FS analysis is that $\rho(D_F \mid B) = \delta(F - 4)$. But this conclusion is unacceptable, in view of the examination above of the probability distribution over locations of the die's center of mass.

Essentially, the reply to Hobson is now finished. However, some rather conjectural remarks may be of use in judging Jaynes' maximum entropy prescription. Consider cases in which the data B do not specify that the successive trials concern the same physical object or even different physical objects in the same environment. By being completely uninformative about the physical relations of the various trials, the data B presumably do satisfy the premiss of "not linking." Then Hobson's theorem applies and its conclusion follows. Is this reasonable? The answer again seems to be no, though perhaps personal judgment has intruded. While the data B do not assert a linkage trials, they do not preclude one; they are simply mute on the question. To justify Hobson's conclusion, one would have to say that on the bare background data B, the probability of the existence of a linking mechanism which would cause a statistical deviation from $\langle f \rangle$ is zero. We would then seem to be saying that, given only B, the phenomena under consideration are with overwhelming probability uncorrelated. But would not such an assertion transgress the pretense of ignorance about the existence or nonexistence of linkages when only the data B are provided?

The proposition that, given only B, the trials are with overwhelming probability unlinked, seems to be implicit in Hobson's application of Jaynes's maximum entropy principle to $\mathscr{S}^n$, since this application correctly leads to $P(i_1 i_2 \cdots i_n \mid B) = \prod P(i_k \mid B)$. Hence, if this application is indeed required by Jaynes' theses, then we have new grounds for challenging the maximum entropy principle. When that principle is enunciated in general terms it is indeed very plausible, though it is not as intuitively compelling as the principles of deductive logic or even of probability theory. Assent to the general principle should reasonably be deferred until its consequences are explored. If the principle entails the result that background data which are mute about linkages suffice for assigning very high probability to the nonexistence of linkages, then the principle itself is highly suspect.

Finally, it should be noted that $P(i_1 i_2 \cdots i_n \mid B) = \prod P(i_k \mid B)$ is yielded only by the confirmation function $c^\dagger$ among the functions studied by Carnap.[4] Hence, Carnap's reasons for rejecting $c^\dagger$ as a tool for inductive reasoning are relevant to Hobson's proposals.

APPENDIX

The notation of FS will be used. Their assumption that for some i, $E_i = \sum_{j=1}^{n} E_j/n$ is dropped, but it will be assumed that the E_j are distinct and that $n \geqslant 3$. Equation (7) of Ref. 2 can be written not only for $j = i$ but for any j, so that

$$1/n = \int e^{-\beta E_j}(e^{-\beta E_1} + \cdots + e^{-\beta E_n})^{-1}\, dF(d_\beta \mid b) \qquad \text{for each} \quad j$$

Consider the function $g(E)$ which results from the right-hand side of the preceding equation when E_j is replaced by E. It is then easily seen that dg/dE increases monotonically with E unless the probability distribution over d_β

is concentrated at $\beta = 0$. But if dg/dE is monotonically increasing, then $g(E)$ cannot have the same value at three (or more) values of E. Since $g(E_j) = 1/n$ for each j, it then follows that $\rho(d_\beta \mid b) = \delta(\beta)$. In Hobson's notation this is equivalent to $\rho(D_F \mid B) = \delta(D_F - \langle f \rangle)$.

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